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The superconducting half-space with a vortex: closed form results

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Abstract. The fundamental axisymmetric boundary value problem of a single vortex penetrating a half-space of a type-II superconductor is investigated. From the appropriate static solution, closed form results are developed for all electromagnetic fields and densities within the superconductor. For the half-space above the superconductor, closed form results are obtained for all quantities with the introduction of a sole Laplace transformation. These results provide the analytic evaluation of a wide variety of derived quantities including magnetic moment, energy, and flux. The discussion has specific applications to magnetic force microscopy and other probe techniques.

Recently there have been many advances in surface and scanning probe technologies [1]. In particular, low-temperature magnetic force microscopy has been applied to the imaging of superconductors [2, 3]. In this technique, a flexible cantilever is used to measure the magnetic force between a magnetized tip and the field of the sample. Force detection at the picoNewton level is possible and may be improvable. It is possible that refined MFM or similar techniques may be able to shed light on the underlying mechanism in high transition temperature (T_c) materials by precise measurements of the penetration depth. Other information that may be derivable from MFM measurements is the nature of vortex pinning in many sorts of materials.

One of the fundamental theoretical problems in this area is determining the electromagnetic fields within and without a superconducting half-space. Here the sample is treated as semi-infinite, with a perfectly flat surface. This problem has been approached by several different means, and an early significant calculation was that of Pearl [4, 5], to which I briefly return near the end. Lately it has been shown how the magnetic field, which is of primary interest to MFM, can be solved for directly [6, 7]. The appropriate interface conditions and Greens functions have been written [7]. However, previous treatments have been satisfied with integral representations of the results, often only giving explicit results in limiting cases. This paper develops closed form results, showing the incompleteness of earlier work, and how numerical integration can be avoided in most cases.

I make the assumption of axisymmetry throughout this exposition, and rely on background given in [7]. I consider the problem of a single vortex perpendicularly penetrating a half-space. It is shown that the fields and densities within the superconductor (for $z \leqslant 0$) can be put into closed form and the field above the superconductor requires the introduction of only one Bessel function integral.

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The subscript 1 emphasizes upper half-space and 2 lower half-space quantities [7]. Let λ_2 be the London penetration depth and ϕ_0 the flux quantum. Within London theory the magnetic field may be determined by Fourier transformation and is given by [7]

$$H_{1z}(\rho, z \geqslant 0) = \frac{\phi_0}{2\pi\lambda_2^2}\int_0^\infty \frac{k\,\mathrm{d}k}{\gamma_2(k+\gamma_2)} J_0(k\rho)\mathrm{e}^{-kz} \tag{1}$$

$$H_{2z}(\rho, z \leqslant 0) = -\frac{\phi_0}{2\pi\lambda_2^2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{\gamma_2^2(k+\gamma_2)} J_0(k\rho)\mathrm{e}^{\gamma_2 z} + \frac{\phi_0}{2\pi\lambda_2^2} K_0\left(\frac{\rho}{\lambda_2}\right) \tag{2}$$

$$H_{1\rho}(\rho, z \geqslant 0) = \frac{\phi_0}{2\pi\lambda_2^2}\int_0^\infty \frac{k\,\mathrm{d}k}{\gamma_2(k+\gamma_2)} J_1(k\rho)\mathrm{e}^{-kz} \tag{3}$$

$$H_{2\rho}(\rho, z \leqslant 0) = \frac{\phi_0}{2\pi\lambda_2^2}\int_0^\infty \frac{k\,\mathrm{d}k}{\gamma_2(k+\gamma_2)} J_1(k\rho)\mathrm{e}^{-\gamma_2 z} \tag{4}$$

and the supercurrent density by

$$j_{s\theta 2}(\rho, z \leqslant 0) = \frac{c\phi_0}{8\pi^2\lambda_2^4}\int_0^\infty \frac{\mathrm{d}k\, J_1(k\rho)k\mathrm{e}^{\gamma_2 z}}{(k^2+\lambda_2^{-2})(k+\gamma_2)} + \frac{c\phi_0}{8\pi^2\lambda_2^3} K_1\left(\frac{\rho}{\lambda_2}\right) \tag{5}$$

where $\gamma_2(k) \equiv \sqrt{k^2+\lambda_2^{-2}}$. In equations (1)–(5) J_n is the Bessel function of order n; K_0 and K_1 are order zero and one modified Bessel functions, respectively.

Of special interest are the values of interface quantities, those at $z = 0$. These are simpler expressions which should be useful for very-near surface scans in probe microscopy. They also provide insightful special cases of the later results. The magnetic field at the surface is given by

$$\frac{2\pi\lambda_2^2}{\phi_0} H_z(\rho \neq 0, z = 0) = \frac{1}{2}\left[I_0\left(\frac{\rho}{2\lambda_2}\right)K_0\left(\frac{\rho}{2\lambda_2}\right) - I_1\left(\frac{\rho}{2\lambda_2}\right)K_1\left(\frac{\rho}{2\lambda_2}\right)\right] \tag{6}$$

and

$$H_\rho(\rho, z = 0) = \frac{\phi_0}{2\pi\rho}\left[\frac{1}{\rho} - \left(\frac{1}{\rho}+\frac{1}{\lambda_2}\right)\mathrm{e}^{-\rho/\lambda_2}\right]. \tag{7}$$

The modified Bessel functions of order n are I_n and K_n of the first and second kind, respectively. In equation (6), when $\rho = 0$, it is necessary to add a term $(\lambda_2^2/\rho)\delta(\rho)$ to the right-hand side, where $\delta(\rho)$ is a one-dimensional Dirac delta function. The supercurrent density at the surface is

$$j_{s\theta}(\rho, z = 0) = -\frac{c\phi_0}{16\pi^2\lambda_2^3}\left[I_1\left(\frac{\rho}{2\lambda_2}\right)K_0\left(\frac{\rho}{2\lambda_2}\right) - I_0\left(\frac{\rho}{2\lambda_2}\right)K_1\left(\frac{\rho}{2\lambda_2}\right)\right]. \tag{8}$$

In order to give an indication of how the results (6)–(8) may be obtained, note from equations (3) or (4) that the radial component of magnetic field may be written as [8]

$$\begin{aligned} H_\rho(\rho, z = 0) &= \frac{\phi_0}{2\pi}\frac{\partial}{\partial\rho}\int_0^\infty \frac{\mathrm{d}k\, J_0(k\rho)}{\sqrt{k^2+\lambda_2^{-2}}}\left(k - \sqrt{k^2+\lambda_2^{-2}}\right) \\ &= \frac{\phi_0}{2\pi}\frac{\partial}{\partial\rho}\frac{1}{\rho}\left(\mathrm{e}^{-\rho/\lambda_2} - 1\right). \end{aligned} \tag{9}$$

For the vertical component it may be noted from equation (1) that

$$\begin{aligned} \frac{2\pi\lambda_2^2}{\phi_0} H_z(\rho, z = 0) &= -\int_0^\infty \frac{\mathrm{d}k' k'}{\sqrt{1+k'^2}}\left(k' - \sqrt{1+k'^2}\right) J_0\left(\frac{\rho}{\lambda_2}k'\right) \\ &= -\lambda_2^2\int_0^\infty \frac{\mathrm{d}k\, k^2}{\sqrt{k^2+\lambda_2^{-2}}} J_0(k\rho) + \frac{\lambda_2^2}{\rho}\delta(\rho). \end{aligned} \tag{10}$$

The delta function arises as the result of the orthogonality of Bessel functions on the half line. By using the Bessel differential equation for J_0, the first term on the right-hand side of equation (10) can be written as

$$\frac{\lambda_2^2}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}I_0\left(\frac{\rho}{2\lambda_2}\right)K_0\left(\frac{\rho}{2\lambda_2}\right). \tag{11}$$

Evaluating the partial derivatives with respect to ρ, using the derivatives $K_0' = -K_1$ and $I_0' = I_1$, and using the Wronskian relation

$$I_1(u)K_0(u) + I_0(u)K_1(u) = \frac{1}{u} \tag{12}$$

then gives equation (6).

For purposes of describing the fields in the superconductor, I introduce the definite integrals

$$N(\rho,z) \equiv \int_0^\infty \frac{e^{z\gamma_2}}{\gamma_2}J_0(k\rho)\,dk = I_0\left(\frac{R+z}{2\lambda_2}\right)K_0\left(\frac{R-z}{2\lambda_2}\right) \tag{13a}$$

$$P(\rho,z) \equiv \int_0^\infty \frac{e^{z\gamma_2}}{\gamma_2}kJ_0(k\rho)\,dk = \frac{e^{-R/\lambda_2}}{R} \tag{13b}$$

where $R^2 = \rho^2 + z^2$. The function $N(R,z)$ is the same as the Foster–Lien integral $L(a,r)$ [7]. Within the superconductor the z-component of the magnetic field is

$$H_{2z}(\rho, z \leqslant 0) = \frac{\phi_0}{2\pi\lambda_2^2}K_0\left(\frac{\rho}{\lambda_2}\right) + \frac{\phi_0}{2\pi}\left(\frac{\partial P}{\partial z} - \frac{\partial^2 N}{\partial z^2} - \lambda_2^{-2}\int_{-\infty}^{z} P\,dz' + \lambda_2^{-2}N\right) \tag{14}$$

while the ρ-component is given by

$$H_{2\rho}(\rho, z \leqslant 0) = \frac{\phi_0}{2\pi}\frac{\partial}{\partial\rho}\left(P - \frac{\partial N}{\partial z}\right). \tag{15}$$

The supercurrent density is given by

$$j_{s\theta 2}(\rho, z \leqslant 0) = \frac{c\phi_0}{8\pi^2\lambda_2^2}\frac{\partial}{\partial\rho}\left(\int_{-\infty}^{z} P\,dz' - N\right) + \frac{c\phi_0}{8\pi^2\lambda_2^3}K_1\left(\frac{\rho}{\lambda_2}\right). \tag{16}$$

It can be easily checked that equations (14)–(16) comply with Ampere's law,

$$j_{s\theta 2} = \frac{c}{4\pi}\left(\frac{\partial H_{2\rho}}{\partial z} - \frac{\partial H_{2z}}{\partial\rho}\right). \tag{17}$$

Carrying out the partial differentiations, the radial component of the magnetic field can be written as

$$H_{2\rho}(\rho, z \leqslant 0) = \frac{\phi_0}{2\pi}\left[-\frac{\rho}{R^2}\left(\frac{1}{R} + \frac{1}{\lambda_2}\right)e^{-R/\lambda_2} - \frac{\partial^2 N}{\partial\rho\partial z}\right] \tag{18a}$$

where

$$\begin{aligned} 2\lambda_2\frac{R}{\rho}\frac{\partial^2 N}{\partial\rho\partial z} = {} & \frac{z}{\lambda_2 R}\left[I_0\left(\frac{R+z}{2\lambda_2}\right)K_0\left(\frac{R-z}{2\lambda_2}\right) - I_1\left(\frac{R+z}{2\lambda_2}\right)K_1\left(\frac{R-z}{2\lambda_2}\right)\right] - \frac{1}{(R+z)} \\ & \times I_1\left(\frac{R+z}{2\lambda_2}\right)K_0\left(\frac{R-z}{2\lambda_2}\right)\left(1 + \frac{z}{R}\right) + \frac{1}{(R-z)}I_0\left(\frac{R+z}{2\lambda_2}\right)K_1\left(\frac{R-z}{2\lambda_2}\right) \\ & \times\left(\frac{z}{R} - 1\right) + \frac{z}{R^2}\left[I_0\left(\frac{R+z}{2\lambda_2}\right)K_1\left(\frac{R-z}{2\lambda_2}\right) - I_1\left(\frac{R+z}{2\lambda_2}\right)K_0\left(\frac{R-z}{2\lambda_2}\right)\right]. \end{aligned} \tag{18b}$$

Equation (15) for $H_{2\rho}$ can be derived by following steps similar to equation (9) and using the definitions (13). In finding the supercurrent density from equation (5), the integral is given by

$$\frac{c\phi_0}{8\pi^2\lambda_2^4}\int_0^\infty \frac{\mathrm{d}k\,k J_1(k\rho)}{(k^2+\lambda_2^{-2})}\frac{\mathrm{e}^{\gamma_2 z}}{\left(k+\sqrt{k^2+\lambda_2^{-2}}\right)} = \frac{c\phi_0}{8\pi^2\lambda_2^2}\frac{\partial}{\partial\rho}\int_0^\infty \frac{\mathrm{d}k\ J_0(k\rho)}{(k^2+\lambda_2^{-2})} \times\mathrm{e}^{\gamma_2 z}\left(k-\sqrt{k^2+\lambda_2^{-2}}\right). \tag{19}$$

In finding H_{2z} from equation (2), the factor $k^2/(k^2+\lambda_2^{-2})$ can be written as $1-\lambda_2^{-2}/(k^2+\lambda_2^{-2})$, giving the integral

$$-\lambda_2^2\int_0^\infty \mathrm{d}k\ J_0(k\rho)\mathrm{e}^{\gamma_2 z}\left(1-\frac{\lambda_2^{-2}}{k^2+\lambda_2^{-2}}\right)\left(k-\sqrt{k^2+\lambda_2^{-2}}\right). \tag{20}$$

In turn this integral can be found in terms of the functions P and N.

For describing the magnetic field in the upper half-space, put

$$M(\rho, z) \equiv \int_0^\infty \frac{J_0(k\rho)}{\gamma_2}\mathrm{e}^{-kz}\mathrm{d}k. \tag{21}$$

This integral can be interpreted variously as either a Laplace or Bessel transform; it is the sole remaining definite integral not described in terms of solutions of the Bessel differential equation. Then the vertical component of magnetic field is from equation (1)

$$H_{1z}(\rho, z \geqslant 0) = \frac{\phi_0}{2\pi}\left(-\frac{\partial^2 M}{\partial z^2} + \frac{z}{(\rho^2+z^2)^{3/2}}\right) \tag{22}$$

and the radial component from equation (3) is given by

$$H_{1\rho}(\rho, z \geqslant 0) = \frac{\phi_0}{2\pi}\left(-\frac{\partial^2 M}{\partial\rho\partial z} + \frac{\rho}{(\rho^2+z^2)^{3/2}}\right). \tag{23}$$

These forms are especially useful for studying the behaviour of the field for large values of z or in the limit $\lambda_2 \to 0$.

From the given expressions for the field components it is then possible to compute magnetic field energy and magnetic fluxes through specified surfaces [6, 7]. As a simple example, the total flux up through the interface $z = 0$ can be computed using equation (6) as

$$2\pi\int_0^\infty \rho\,\mathrm{d}\rho H_z(\rho, z = 0) = \phi_0. \tag{24}$$

In evaluating the integral the net contribution of the modified Bessel functions can be shown to be zero, by integration by parts. The flux quantum arises as the integration over the delta function contribution at $\rho = 0$. Similarly, the total radially outward magnetic flux in the upper half-space is ϕ_0. This can be seen by evaluating the integral $\lim_{\rho\to\infty} 2\pi\rho\int_0^\infty \mathrm{d}z\ H_{1\rho}(\rho, z \geqslant 0)$.

The use of the Wronskian relation (12) shows equation (8) to be equivalent to the interface result obtained by Pearl [4, 5]. The general integral evaluations avoided there have been performed in this paper.

This paper has completed the analytic evaluation of the fields and densities within a semi-infinite superconductor containing a single vortex. London theory has been used wherein a straight Abrikosov vortex is treated as a line source. As usual, Meissner screening within the superconductor causes the fields to decrease with increasing distance. On the other hand, for an analogous problem of wave propagation in a half-space, [9] may be consulted.

The integrals treated here come from a Fourier–Bessel–Laplace representation of fields. The magnetic field and supercurrent density have been explicitly written. From either direct integration of the relation $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ or by use of the London relation the vector potential $\boldsymbol{A}$ can be readily obtained. The only remaining integral for the upper half-space is the function $M(\rho, z)$, equation (21), and it can be written in several different forms [10]. Therefore, the use of numerical quadrature can be greatly minimized.

From the results presented here a wide variety of derived quantities can be calculated, including magnetic moments and fluxes, as mentioned. Furthermore, the magnetostatic interaction energies and forces needed for magnetic force microscopy can be obtained. For a discussion of such results for axisymmetric boundary value problems for a superconductor with a planar surface, see [11]. There closed form expressions are obtained for arbitrary probe height above the vortex.

The closed form results given here may be all the more important for use in magnetic resonance force microscopy (MRFM) [1]. In this technique, a small rf magnetic field is used in addition to, for example, a sample mounted on an elastic cantilever in the inhomogeneous field of a magnetized tip, thus combining features of nuclear magnetic resonance imaging and atomic force microscopy. This method then holds promise for probing the detailed electromagnetic structure of a sample below its surface. If MRFM can be applied to superconductors, then many interesting theoretical and experimental possibilities should evolve.

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[5] In [4] the following typographical errors can be noted. For the numerator of equation (12), $\gamma^{-1/2}$. For the left-hand sides of equations (14) and (15), the factors λ^{-2} and γ^{-1}, respectively. For the denominator of the integrand in equation (20), $(\gamma^2 + \lambda^{-2})^{1/2}$, and $K_1(r/\lambda)$ for the right-hand side of equation (21).

[6] Coffey M W 1995 *Phys. Rev.* B **52** 1187. The line just above equation (35) in this paper should read $g \to 1/3p$ and accordingly the occurrences of $2/3$ factors in the rest of that section are extraneous.

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